Understanding the coefficient of restitution (COR) using mass/spring systems

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The coefficient of restitution (COR) for a solid object such as a baseball, colliding with a perfectly rigid wall can be defined as the ratio of the outgoing speed to the incoming speed. When a hardened steel ball bearing collides with a large hardened steel plate, the collision has a COR of close to one. On the other extreme, when a foam (Nerf[i]™) ball collides with the same plate, the collision has a COR of nearly zero. It seems to be generally the case that flexible objects more often suffer low COR collisions, while rigid objects are more likely to undergo higher COR collisions. The intent here is to explain this behavior with a mass/spring system model.

COR and Elastic Collisions

The COR is related to the conversion of the initial kinetic energy into internal energy during the collision. For the baseball collision described above, the COR would be one if all the initial kinetic energy were conserved and appeared as the kinetic energy of the outgoing ball. You could imagine the COR dropping as some parameter of the ball is varied and the collision repeated. As the COR decreases toward zero, a greater and greater fraction of the initial kinetic energy is converted to internal energy in the ball. So, the COR is one for elastic collisions and less than one for inelastic collisions.

Webster's[1] defines the elastic to mean, "easily stretched or expanded," while synonyms such as "inflexible" and "unyielding" are given for inelastic. It is quite ironic that elastic objects tend to experience inelastic collisions and inelastic objects are more likely to undergo elastic collisions. In fact, it is possible that this disparity between common language and technical jargon is a source of confusion for our students.

The goal of this discussion is to understand conceptually why easily deformed objects tend to have inelastic collisions while inflexible objects tend to have more elastic collisions using the least complex model of a solid as possible.

A Very Simple Model of a Solid

Modeling a solid as a collection of masses and springs can have pedagogic benefits as students have a sense, perhaps from chemical models, that solids are composed of atoms (masses) held in place by electromagnetic forces (springs). Ganiel[2] actually describes a lecture demonstration using a cart carrying masses and springs that illustrates where the "missing" kinetic energy goes in an inelastic collision. Zou[3] improved the design of the cart and shared a series of guided-inquiry learning activities for their use. Other authors have attempted to understand the transfer of mechanical energy into internal energy using mass and spring models[4,5,6].

Reducing the mass and spring model as far as possible leaves two masses, m, connected by a spring of spring constant, k, as shown in Figure 1. The masses are infinitely rigid and have no internal structure, so they can't possess any internal energy. The internal energy in this system is in the vibrations of the spring.

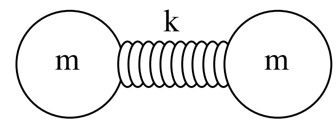

Figure 1: The colliding object is represented by two equal masses, m, connected by a spring of spring constant, k.

Figure 2 illustrates the collision of our object with a perfectly rigid wall, one that doesn't absorb any internal energy. The incoming speed of the object is v_o. Both masses initially move at this speed so the only energy in the object is the kinetic energy of these masses. The total energy of the system is,

$$K_o = \tfrac{1}{2}mv_o^2 + \tfrac{1}{2}mv_o^2 = mv_o^2. \tag{1}$$

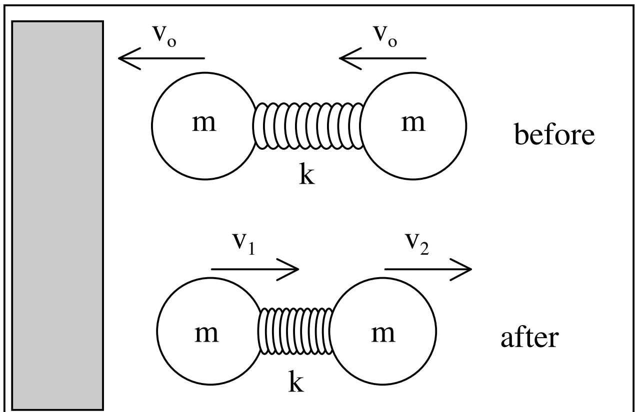

Figure 2: The collision of the object with a perfectly rigid wall.

After the collision, the mass that collides with the wall has a speed v_1 and the other mass has a speed v_2. Now, the speed of the center of mass is $v = \tfrac{1}{2}(v_1 + v_2)$ and the kinetic energy of the center of mass is,

$$K = \tfrac{1}{2}(2m)v^2 = \tfrac{1}{4}m(v_1 + v_2)^2. \tag{2}$$

This kinetic energy of the center of mass is equivalent to the macroscopic kinetic energy of the object. The remainder of the energy goes into the internal energy associated with the oscillations of the mass/spring system,

$$U_{\text{int}} = mv_o^2 - \tfrac{1}{4}m(v_1 + v_2)^2. \tag{3}$$

This energy is equivalent macroscopically to the internal thermal energy of the object. If the macroscopic kinetic energy is conserved, then the collision is macroscopically elastic. That is,

$$U_{\text{int}} = 0 \Rightarrow \text{ elastic collision} \qquad (4)$$
$$U_{\text{int}} > 0 \Rightarrow \text{ inelastic collision.}$$

The COR of the collision is the ratio of the center of mass velocity after the collision to the center of mass velocity before,

$$COR = \frac{v_{cm,f}}{v_{cm,i}} = \frac{\frac{1}{2}(v_1 + v_2)}{\frac{1}{2}(v_o + v_o)} = \frac{v_1 + v_2}{2v_o}. \qquad (5)$$

The COR can be related directly to the kinetic energies,

$$COR = \sqrt{\frac{K}{K_o}}. \qquad (6)$$

Therefore, the COR is one for elastic collisions and as the COR gets smaller, more and more of the initial kinetic energy is converted to the internal energy in the spring/mass system resulting in a COR less than one. For the remainder of this discussion we will focus on the COR keeping in mind that it is a surrogate for the energy transferred to internal energy.

A Naïve Model of a Collision

In our very simple model of a solid, the only available parameter to vary is the spring constant. A simple thought is that a very large spring constant models a very rigid object, while a very small spring constant models a very flexible object. We can then look for variations in the COR due to the rigidity of the simple solid. The mass/spring system heads toward the wall as shown in Figure 3a. When the mass first collides with the wall, it instantaneously reverses direction and keeps the same velocity, since neither the wall nor the mass can absorb any internal energy.

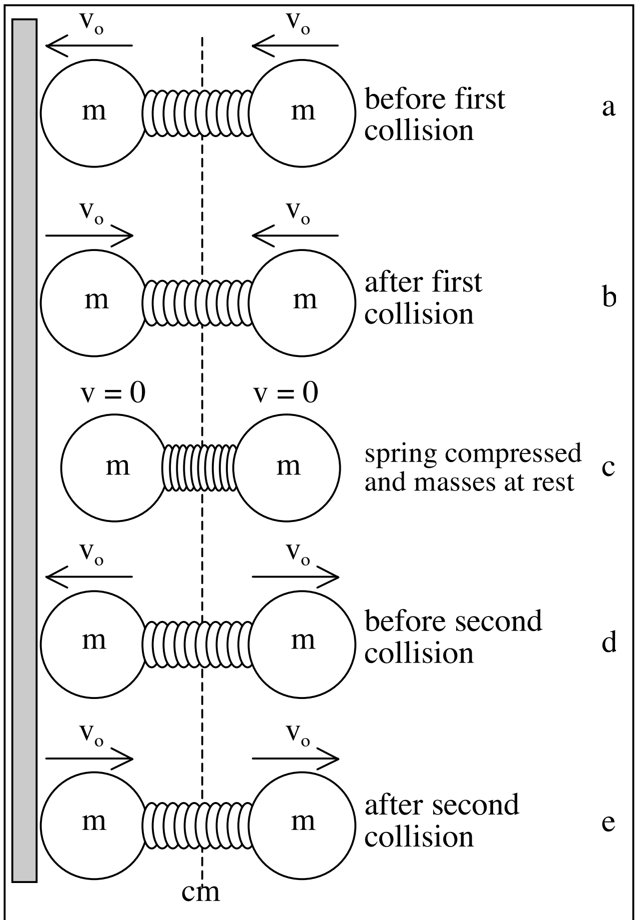

Figure 3: Two collisions of the mass/spring system with a perfectly rigid wall. Note that the center of mass remains fixed from the time of the first collision until the end of the second collision.

Now the two masses are heading toward each other at the same speed with the uncompressed spring between them as in Figure 3b. Notice that the mass/spring system has no center of mass velocity. Soon, both masses are at rest with the compressed spring between them as in Figure 3c. The masses now reverse direction and begin to speed up. All the while, the center of mass has not moved. When the mass is just about to strike the wall again, it is traveling at the same speed it had before the first collision. Meanwhile the other mass has the same speed as before, but has reversed direction as shown in Figure 3d. The spring is now uncompressed. The mass collides with the wall and reverses direction again. Now, both masses are headed away from the wall with the same speed as they arrived and the spring is uncompressed between them as in Figure 3e.

The COR for this collision appears to be zero after the first bounce off the wall because the center of mass of the system in Figures 3b, 3c, and 3d is at rest and all of the energy is in the internal motion of the mass/spring system. However, after the second bounce, the COR is one and the collision is elastic regardless of the spring constant. Our naïve idea the rigidity of an object can be modeled by varying the internal spring constant seems doomed[7]. However, the impulse exerted by the wall on the first mass during the collision is actually far more complex than the instantaneous impulse we have assumed. The "devil is in the details" of the collision which can be modeled using the speed of compressional waves that travel along the spring. Roura[8] presents a very clean conceptual discussion of this issue.

Toward a Better Model

A way to simplify the intricacies of the collision with the wall is to just use a second spring to moderate it. Cross[9] applies this model to explain the famous basketball and tennis ball demo. In our case, two equal masses, m, joined by an internal spring of spring constant, k, collide with a solid massive wall. The collision is mediated by a second external spring of spring constant, βk, which can only exert forces in compression and not in extension. The springs both have the same natural length, a, as shown in Figure 4. Large values of β model a solid with weak springs or one that is fairly flexible, while small values of β suggest a solid with strong internal springs or one that is rather rigid.

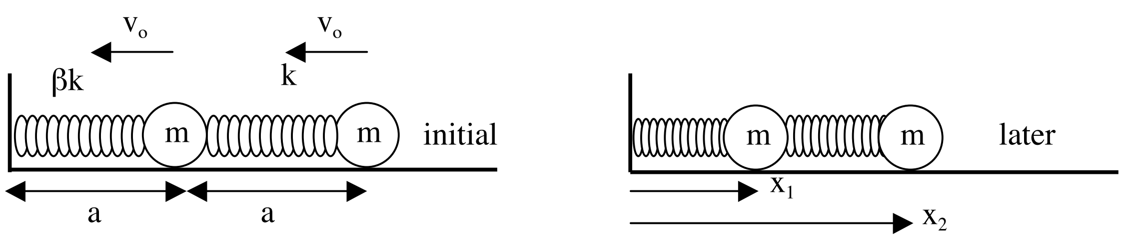

Figure 4: Two equal masses, m, joined by a spring of spring constant, k, and natural length, a, collide with a solid massive wall. The collision is mediated by a second spring of spring constant, βk.

The goal is to follow the motion of the system until it stops interacting with the wall ($x_1 > a$). Then find the center of mass speed of the system to calculate the COR for the collision. Writing Newton's Second Law for each mass when $x_1 < a$,

$$m\frac{d^2x_1}{dt^2} = \beta k(a - x_1) - k[a - (x_2 - x_1)] \tag{7}$$

$$m\frac{d^2x_2}{dt^2} = +k[a - (x_2 - x_1)]. \tag{8}$$

Switching to the dimensionless variables,

$$u_1 \equiv 1 - \tfrac{x_1}{a},\ u_2 \equiv 1 - (\tfrac{x_2}{a} - \tfrac{x_1}{a}), \text{ and } \tau \equiv \sqrt{\tfrac{k}{m}}t. \tag{9}$$

Noting that,

$$x_1 = a(1 - u_1), \tag{10}$$

$$x_2 = a(1 - u_2) + x_1 = a(2 - u_2 - u_1), \tag{11}$$

$$\frac{dx_1}{dt} = \frac{dx_1}{du_1}\frac{d\tau}{dt}\frac{du_1}{d\tau} = -a\sqrt{\tfrac{k}{m}}\frac{du_1}{d\tau} \Rightarrow \frac{d^2x_1}{dt^2} = -a\frac{k}{m}\frac{d^2u_1}{d\tau^2} = -a\frac{k}{m}u_1'' \tag{12}$$

$$\frac{dx_2}{dt} = \frac{dx_1}{dt} - a\frac{du_2}{dt} = -a\sqrt{\tfrac{k}{m}}\frac{du_1}{d\tau} - a\sqrt{\tfrac{k}{m}}\frac{du_2}{d\tau} = -a\sqrt{\tfrac{k}{m}}\left(\frac{du_1}{d\tau} + \frac{du_2}{d\tau}\right)$$

$$\frac{d^2x_2}{dt^2} = -a\frac{k}{m}\left(\frac{d^2u_1}{d\tau^2} + \frac{d^2u_2}{d\tau^2}\right) = -a\frac{k}{m}\left(u_1'' + u_2''\right). \tag{13}$$

The Second Law equations become,

$$u_1'' + \beta u_1 - u_2 = 0 \text{ and } u_1'' + u_2'' + u_2 = 0. \tag{14}$$

When the system is not interacting with the wall ($x_1 > a$), the Second Law equations are the same except β = 0. Equations 7 and 8 are subject to the initial conditions,

$$x_1(0) = a,\ x_2(0) = 2a,\ \dot{x}_1(0) = -v_o, \text{ and } \dot{x}_2(0) = -v_o, \tag{15}$$

which means,

$$u_1(0) = 0,\ u_2(0) = 0,\ u_1'(0) = \alpha, \text{ and } u_2'(0) = 0, \tag{16}$$

where,

$$\alpha \equiv \sqrt{\tfrac{mv_o^2}{ka^2}}. \tag{17}$$

You can solve these equations by the method of normal modes[10] if you are enamored with elegance, closed form solutions, and advanced mathematics. If you are less skilled and willing to tolerate numerical round off errors, you can use a spreadsheet. I naturally chose the later.

By varying the value of β, you can go from a very gentle collision with a stout internal spring (small β) to a very abrupt collision (large β) with a soft internal spring. The data from the calculations are summarized in Table 1. The second column is the COR after the first bounce. For large β, the COR is small after the first bounce, exactly as described earlier for the mass/spring system striking the rigid wall. After one bounce, all the energy is in the internal motion of the mass/spring system as in Figures 3b, c, and d. The second column is the COR after the second bounce. For large β, the COR is one after the second bounce, in agreement with the earlier analysis of Figure 3e. The motion of each mass as a function of time for β = 10 is shown in Figure 5. Notice that there is very little oscillatory motion as the mass/spring system leaves the wall consistent with a COR near one.

For small β the COR is also equal to one. The case of β = 0.1 is also shown in Figure 5. Notice again there is almost no oscillatory motion as the mass/spring system exits. However, for intermediate values of β the COR is noticeably less than one. In Figure 5 the graph for β=2 shown illustrating a modest amount of oscillatory motion after the interaction with the wall because the COR has now dropped below one.

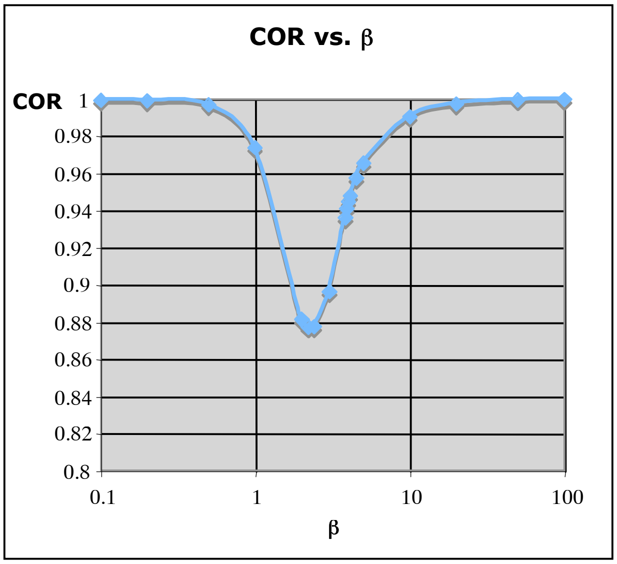

Figure 6: COR vs. β for the two mass and spring system.

β	COR1	COR2	COR
100	0.005	1.000	1.000
50	0.010	1.000	1.000
20	0.025	0.997	0.997
10	0.052	0.990	0.990
5.0	0.119	0.965	0.965
4.5	0.138	0.957	0.957
4.1	0.159	0.948	0.948
4.0	0.165	0.945	0.945
3.9	0.173	0.941	0.941
3.8	0.936		0.936
3.0	0.896		0.896
2.4	0.878		0.878
2.2	0.877		0.877
2.0	0.882		0.882
1.0	0.974		0.974
0.5	0.997		0.997
0.2	0.999		0.999
0.1	0.999		0.999

Table 1: The COR as a function of β after each bounce and total.

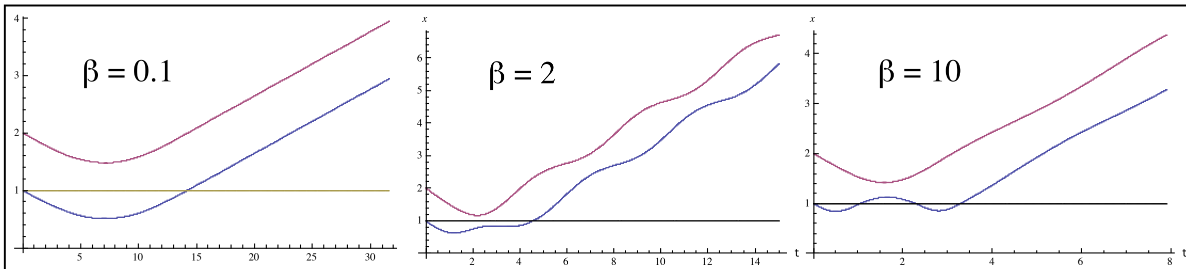

Figure 5: Position vs. time graphs for each mass. The blue line tracks the mass that interacts with the wall via the external spring while the red line follows the motion of the other mass. The horizontal line is at x = a where the external spring stops acting on the mass/spring system.

A graph of COR vs. β is the subject of Figure 6. While the COR is one for both large and small values of β, it drops at intermediate values with a minimum around β = 2. Once again our simple idea that the rigidity of an object can be modeled by varying the internal spring constant appears to be thwarted.

The shape of Figure 6 suggests that we are looking at a resonance phenomenon. The internal spring has a natural oscillation as illustrated in Figures 3b, c, and d. Since the cm is fixed, the frequency will be the same as that of a single mass oscillating on a spring of half the length and therefore, twice the spring constant so,

$$\omega_{\text{int}} = \sqrt{\tfrac{2k}{m}}. \qquad (18)$$

The external spring, which has a spring constant of βk is acting on the system which has a mass of 2m. It has a natural frequency of,

$$\omega_{ext} = \sqrt{\tfrac{\beta k}{2m}}. \qquad (19)$$

The most energy will be coupled into the internal spring when it is driven at its natural frequency by the external spring,

$$\omega_{ext} = \omega_{\text{int}} \Rightarrow \sqrt{\tfrac{\beta k}{2m}} = \sqrt{\tfrac{2k}{m}} \Rightarrow \beta = 4. \qquad (20)$$

Of course, you will immediately notice that the minimum is actually closer to β=2 than β=4. Looking at the graphs in Figure 5, you will see that it is only for low values of β that the motion of the external spring is close to simple harmonic. For higher values of β the motion is more complex and therefore so is the force exerted by the external spring on the mass/spring system. The mass/spring system could be treated as a forced harmonic oscillator.

The Forced Harmonic Oscillator

To treat the mass/spring system as a forced oscillator let's go back to the Second Law equations and replace the external spring with a time dependent force,

$$m\frac{d^2x_1}{dt^2} = F(t) - k[a-(x_2-x_1)] \qquad (21)$$

$$m\frac{d^2x_2}{dt^2} = +k[a-(x_2-x_1)]. \qquad (22)$$

It is now appropriate to switch to dimensionless variables that represent the center of mass motion and the internal separation of the masses,

$$y_{cm} \equiv \tfrac{x_1+x_2}{2a} = \tfrac{x_{cm}}{a},\ y_{rel} \equiv 1-(\tfrac{x_2}{a}-\tfrac{x_1}{a}) = 1-\tfrac{\Delta x}{a}, \text{ and } \tau \equiv \sqrt{\tfrac{k}{m}}t. \qquad (23)$$

The Second Law equations become,

$$y''_{cm} + \tfrac{1}{2}y''_{rel} = f(\tau) - y_{rel} \text{ and} \qquad (24)$$

$$y''_{cm} - \tfrac{1}{2}y''_{rel} = y_{rel}, \qquad (25)$$

where,

$$f(\tau) \equiv \tfrac{1}{ka}F(t). \qquad (26)$$

Taking the sum and difference,

$$y''_{cm} = \tfrac{1}{2}f(\tau) \text{ and} \qquad (27)$$

$$y''_{rel} + 2y_{rel} = f(\tau). \qquad (28)$$

Equation 27 is simply a statement that the force exerted by the wall, $f(\tau)$, divided by the total mass, 2, is equal to the acceleration of the center of mass. Newton would be pleased. Equation 28 is the equation for the forced oscillation of our oscillator, which has a natural frequency of $\sqrt{2}$, consistent with Equation 18. The energy absorbed by the mass/spring system during the collision will appear as oscillations at this frequency in the center of mass frame.

The solution to the traditional forced oscillator problem[11] is,

$$y_{rel} = \sum_n \frac{c_n}{\omega_{\text{int}}^2 - \omega_n^2} e^{i\omega_n \tau}, \tag{29}$$

where the Fourier component of $f(\tau)$ is given by,

$$c_n = \frac{1}{\mathrm{T}} \int_0^{\mathrm{T}} f(\tau) e^{-i\omega_n \tau} d\tau \text{ and } \omega_n = \frac{2\pi n}{\mathrm{T}}. \tag{30}$$

This is getting far too complicated far too quickly, however, we do have a forced harmonic oscillator. It is just that the forcing function has frequency components at many different frequencies and the amplitudes of these components are not as obvious as we had hoped when we developed Equation 20, which assumed a forcing function at only one frequency.

Putting together Equations 7, 21, and 26,

$$F(t) = kaf(\tau) = \beta k(a - x_1) \Rightarrow f(\tau) = \beta\left(1 - \tfrac{x_1}{a}\right). \tag{31}$$

The forcing function doesn't just depend upon the external spring (β), but also on the position of the colliding mass. Due to the coupled nature of the equations of motion, the position of the colliding mass depends upon the internal spring as well as the external spring. Our two mass/one spring system has only one natural frequency. Since the forcing function is a composed of a collection of frequencies, if we allow our system to have more natural resonances, something interesting is bound to happen.

More Masses and Springs

We can create more natural resonances, if we add more masses and springs to our system. So, let the fun begin! Each time we add a spring and a mass to the right hand side we add another coupled equation to the equations of motion. This creates an additional natural frequency, which could be found using the method of normal modes, and increases the complexity of the forcing function. Since we don't actually need the normal mode frequencies anyway (we just need to understand that they exist) and we have been able to use a spreadsheet to this point, let's just add more columns and madly calculate. The result is the COR versus β curve shown in Figure 7.

Each added natural frequency allows the mass/spring system to absorb more internal energy because of the added natural frequency. In the three mass system you can begin to see that the COR for high β is not longer tending toward one. By the time you get to four masses, the strongly suggestive single resonance shape of the two mass system is entirely gone. At last, we begin to see the general trend we have longed for, rigid low β objects tending to have higher COR's than flexible high β objects.

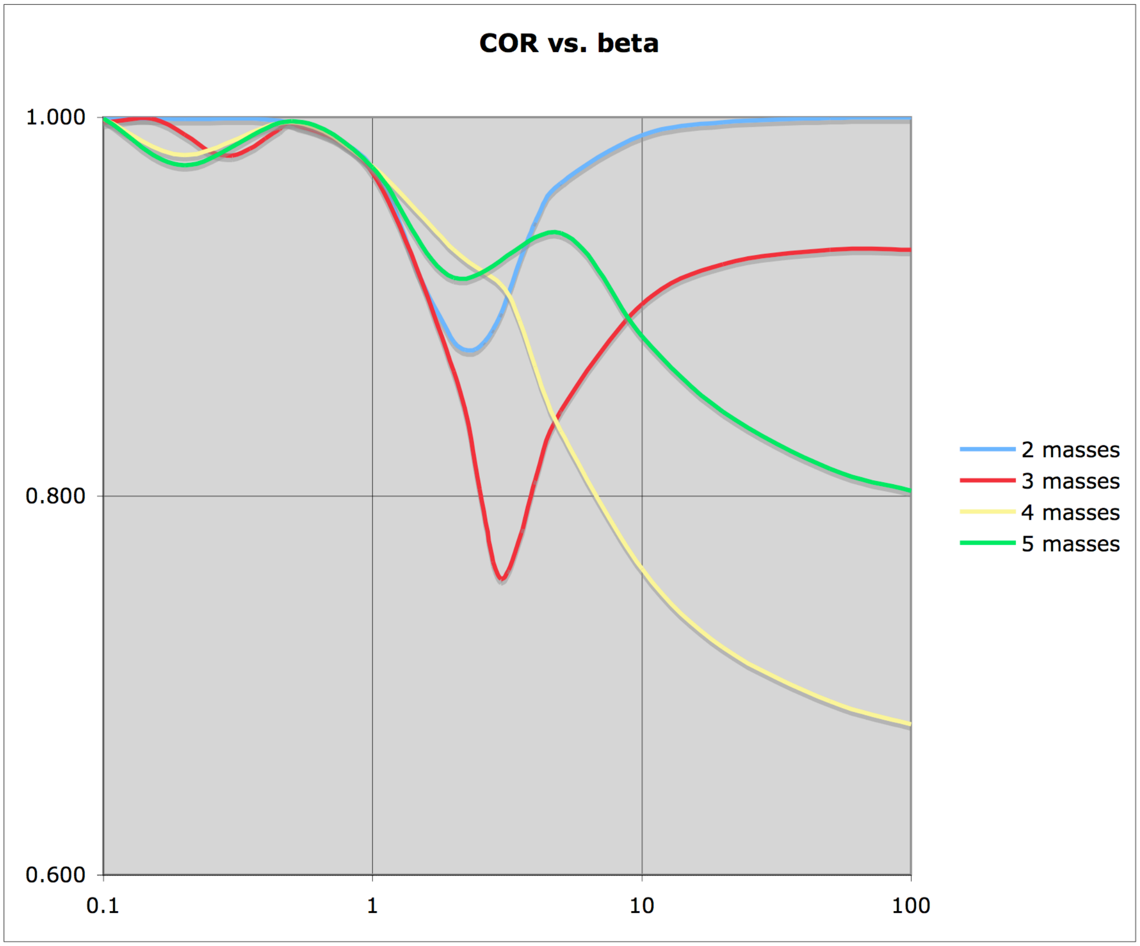

Figure 7: COR versus b for 2 masses (blue), 3 masses (red), 4 masses (yellow), and 5 masses (green).

Higher Octaves

In a sense, we have shown that an object in a collision is like a sophisticated bell, a bell that can produce many distinct tones, one for each natural resonance frequency. The external spring that excites the tones is really the bell itself. During the collision, the object is deformed[12]. This deformation causes compressions in its internal springs. As the collision proceeds, the internal springs can transmit energy to other internal springs if the natural frequencies match the excitation frequency reasonably well.

The chances of a good match depend upon the spacing of the natural frequencies. The closer they are together the better the chances of excitation. The spacing of the natural frequencies depends upon the stiffness of the object. As a basic example, recall[13] the natural oscillations of a string of length L. They are associated with wavelengths of,

$$\lambda_n = \tfrac{2L}{n} \text{ where n = 1, 2, 3, …} \qquad (32)$$

The natural frequencies are,

$$f_n = \frac{v}{\lambda_n} = \frac{nv}{2L}, \qquad (33)$$

and the spacing between them are,

$$f_{n+1} - f_n = \frac{(n+1)v}{2L} - \frac{nv}{2L} \Rightarrow \Delta f = \frac{v}{2L}. \qquad (34)$$

The speed of wave on a string is the square root of the ratio of the tension, F_t, to the mass per unit length, μ, so,

$$\Delta f = \frac{1}{2L}\sqrt{\frac{F_t}{\mu}}. \qquad (35)$$

A more taut string has a higher tension and so more broadly spaced natural frequencies.

The argument presented here for a string holds more broadly. Objects that are more rigid have a greater spacing between their natural frequencies and therefore, these frequencies are less likely to get excited during a collision. It is harder to ring their bell. So as a general rule, stiffer more rigid objects have a higher COR and more flexible object have a lower COR.

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