Derivation of the Exchange of Velocities
For a Perfectly Elastic Collision Between Balls of the Same Mass
Variables

\( m \): The mass of a ball
\( v_{1i} \): Velocity of ball 1 before collision
\( v_{2i} \): Velocity of ball 2 before collision
\( v_{1f} \): Velocity of ball 1 after collision
\( v_{2f} \): Velocity of ball 2 after collision
Consider the elastic collision of two balls of equal mass.

Before Collision

Momentum Before: $mv_{1i}$

Kinetic Energy Before: $\frac{1}{2}mv_{1i}^2$
Consider the motion of the balls after the collision

After Collision

Momentum After: $mv_{1f} + mv_{2f}$

Kinetic Energy After: $\frac{1}{2}mv_{1f}^2 + \frac{1}{2}mv_{2f}^2$
In any collision, momentum is conserved

\[ m v_{1i} = m v_{1f} + m v_{2f} \]

In an elastic collision, Kinetic Energy is conserved

\[ \frac{1}{2} m v_{1i}^2 = \frac{1}{2} m v_{1f}^2 + \frac{1}{2} m v_{2f}^2 \]
Solve for $v_{1f}$ and $v_{2f}$

$$ mv_{1i} = mv_{1f} + mv_{2f} $$
$$ \frac{1}{2} mv_{1i}^2 = \frac{1}{2} mv_{1f}^2 + \frac{1}{2} mv_{2f}^2 $$

(mv conservation)

$$ v_{1i} = v_{1f} + v_{2f} $$

$$ v_{1i}^2 = v_{1f}^2 + v_{2f}^2 $$

(energy conservation)

$$ v_{1i} = v_{1f} + v_{2f} $$

$$ v_{1i}^2 = (v_{1f} + v_{2f})^2 $$

$$ = v_{1f}^2 + 2v_{1f}v_{2f} + v_{2f}^2 $$
but since . . .

\[ v_{1i}^2 = v_{1f}^2 + v_{2f}^2 = v_{1f}^2 + 2v_{1f}v_{2f} + v_{2f}^2 \]

means

\[ 2v_{1f}v_{2f} = 0 \]

so that either

\[ v_{1f} = 0 \quad \text{or} \quad v_{2f} = 0 \]
Since the second ball cannot have zero velocity after the collision, we see that
\[ v_{2f} \neq 0 \]
Therefore \[ v_{1f} = 0 \]
\[ v_{1i} = v_{1f} + v_{2f} \]
\[ v_{1i} = 0 + v_{2f} \]
\[ v_{1i} = v_{2f} \]
Since the balls have the same mass, the velocity (and energy) is transferred from the first ball to the second—that is, they “exchange” velocities. When the balls have different masses, the situation becomes more complicated!