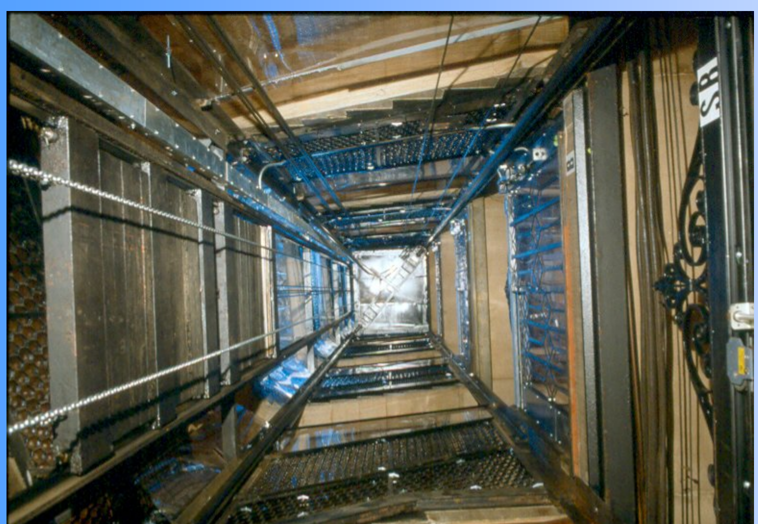
Orbiting Satellites and Free-Fall Elevators



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Dig deep . . .

Suppose you could bore a tunnel through the center of the earth. Further suppose you could pump all the air out of this tunnel to eliminate air friction. What would happen if you devised an elevator that dropped all the through to the other side? This would be one heck of ride. Such an elevator would be like an 8,000-mile *Drop Zone* at Great America!

How long? How fast?

- How long would it take for you to reach the other side of the earth? How long would a round trip be? And how fast would you end up going at the center of the earth?

Round Trip Time = Period

It turns out the round trip time of the elevator is exactly the same time it takes a satellite to orbit the earth—about 90 minutes! This means it would take the elevator 45 minutes to reach the other side of the earth—an impressive feat considering it required no fuel! Why is the time (or period) of the elevator the same as an orbiting satellite?

Same period—a coincidence?

Since the attractive force on the elevator is proportional to the distance from the center of the earth (much the same as the force on a mass suspended on a spring is proportional to the distance displaced) . . .

. . . or *not* a coincidence!

. . . the equations of simple harmonic motion (SHM) apply to *both* the free-fall elevator and the satellite. The period for each is the same.

Newton's 2nd Law and Newton's Law of Gravitation

$$F_{net} = mg = \frac{GMm}{r^2}$$

g inside the earth, *r*
g at the surface of the earth, *R*

Assuming the earth is a uniform sphere--

(1) $$g' = \frac{GM'}{r^2}$$

(2) $$g = \frac{GM}{R^2}$$

Likewise, the acceleration of gravity on the satellite that falls through the hole in the earth varies directly as the distance from the center of the earth, *r*.

(3) $$g' = G\rho\left(\frac{4}{3}\pi r\right)$$

(4) $$g = G\rho\left(\frac{4}{3}\pi R\right)$$

The ratio of *g*'s is the ratio of the radii:

$$\frac{g'}{g} = \frac{G\rho\left(\frac{4}{3}\pi r\right)}{G\rho\left(\frac{4}{3}\pi R\right)}$$

(5) $$\frac{g'}{g} = \frac{r}{R}$$

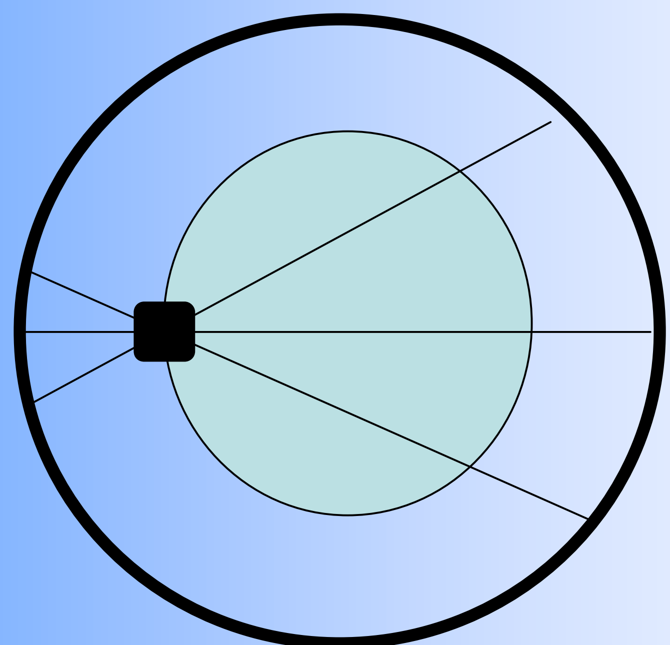
Force inside the earth

(5) $$g' = g\left(\frac{r}{R}\right)$$

$$F = mg'$$

$$F = mg\left(\frac{r}{R}\right)$$

The force on the mass caused by any shell can be found by summing the forces caused by each arc centered on the diameter of the shell.

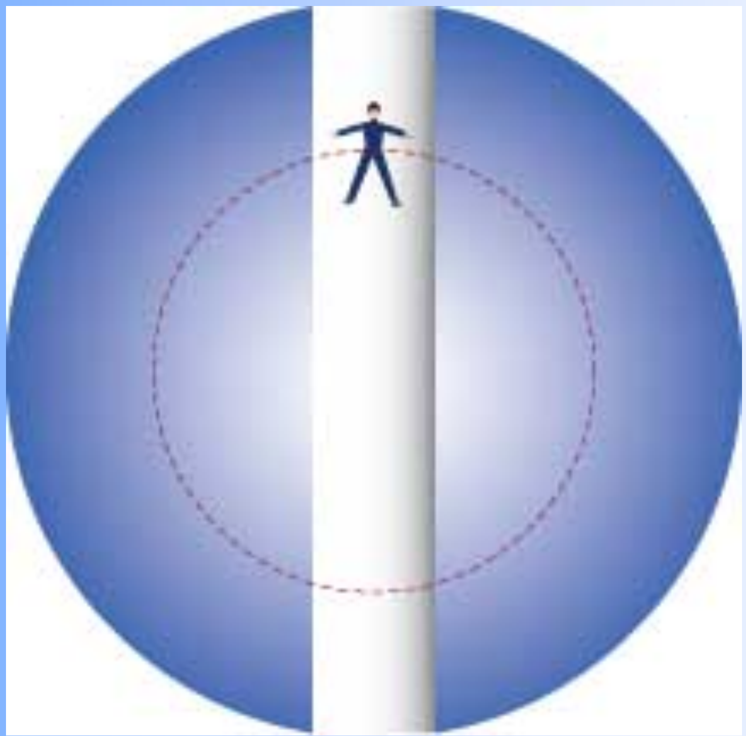


Perfect Cancellation

The force caused by the larger arc is exactly the opposite direction to the force caused by the smaller arc so they tend to cancel. They exactly cancel because the mass in each arc is proportional r^2 while the force caused by the mass is proportional to $1/ r^2$.

The mass that attracts you is the mass below your feet.

Thus, only the mass closer to the center of the earth attracts the object towards the center.



The mass that attracts *you*.

Now lets see how much this closer mass attracts something inside the earth. This is how much it would pull on a person in an elevator shaft that extends through the earth.

The mass enclosed at radius *r* below the earth's surface is:

$$M' = \frac{4}{3}\pi r^3$$

$$M' = \rho V'$$

$$M' = \rho\left(\frac{4}{3}\pi r^3\right)$$

g inside the earth

$$g' = \frac{GM'}{r^2}$$

$$= \frac{G\rho V'}{r^2}$$

$$= \frac{G\rho\left(\frac{4}{3}\pi r^3\right)}{r^2}$$

Force inside the earth

(6) $$F = \left(\frac{mg}{R} \right) r$$

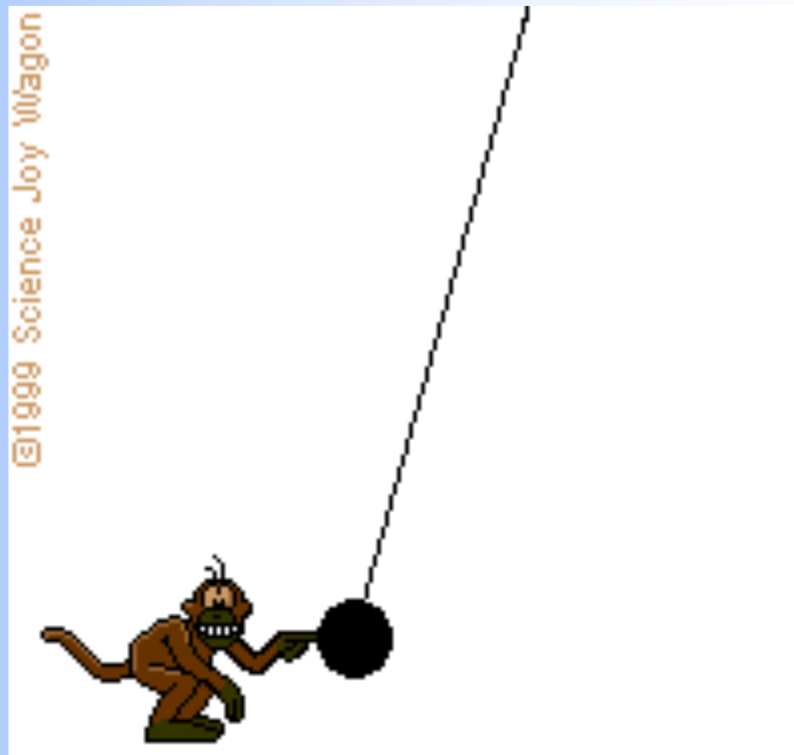
$$F = -kx$$

(7) The condition for SHM is when the acceleration (and hence the force) is proportional to the displacement from the equilibrium position. For the satellite falling through the earth the displacement is *r* and $k = mg/R$.

$$T = 2\pi\sqrt{\frac{m}{k}}$$

$$T = 2\pi\sqrt{\frac{m}{mg/R}}$$

$$T = 2\pi\sqrt{\frac{R}{g}}$$



Do the numbers check?

Substituting the values for $R = 6830$ km and $g = 9.8\ \text{m/s}^2$ . . .

Yes--they agree favorably!

This discrepancy is due to the fact that orbiting satellites are about 250 km above the earth's surface where the value of *g* is about 8% less than it is at the earth's surface.

(8) For a satellite in orbit . . .

$$\frac{mv^2}{R} = mg$$

$$v = \sqrt{Rg}$$

The speed of the satellite is *how much*?

How fast is that?

- Answer: *Pretty damn fast!*

- Substituting the value of R = 6380 km and $g = 9.8$ m/s^2 yields a speed of 7907 m/s or about 8 km/s. This is *fast*—about 17,500 miles per hour!

(9) The speed of a satellite

$$v = \frac{2\pi R}{T}$$

$$\sqrt{Rg} = \frac{2\pi R}{T}$$

(10) Period of a satellite

$$T = \frac{2\pi R}{\sqrt{Rg}}$$

$$T = 2\pi\sqrt{\frac{R}{g}}$$

What about a free-falling elevator?

If we take the PE at the center of the earth to be zero, the PE at the surface equals the KE. However, the PE at the surface is not simply *mgR*. The work needed to lift a mass *m* from the center to the surface is not mg time *R* because the force is not constant—it varies directly with as *r*. Therefore, the it varies from zero at the center to mg at the surface—just like a spring force varies from zero to *kx* when stretched. Because the force is proportional to *r* (like a spring is proportional to *x*) the average force is *mg*/2 and the work is 1/2*mgR*.

$$W = \frac{1}{2} mgR$$

Speed of a free-fall elevator

$$W = PE$$

$$W = \bar{F}d$$

$$= \left(\frac{mg}{2}\right)R$$

Conservation of Energy

$$\Delta PE = -\Delta KE$$

$$\left(\frac{mg}{2}\right)R = \frac{1}{2}mv^2$$

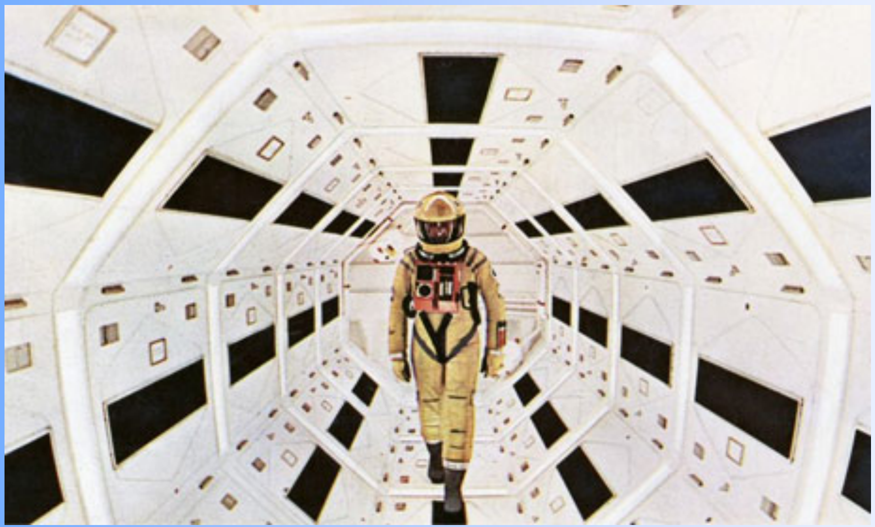
(11) Speed of the free-fall elevator at the center of the earth . . .

$$v = \sqrt{Rg}$$

Amazing!--the same result as a satellite in orbit!

What if?

- What if the tunnel does not go through the center of the earth?
- What is the round-trip time?

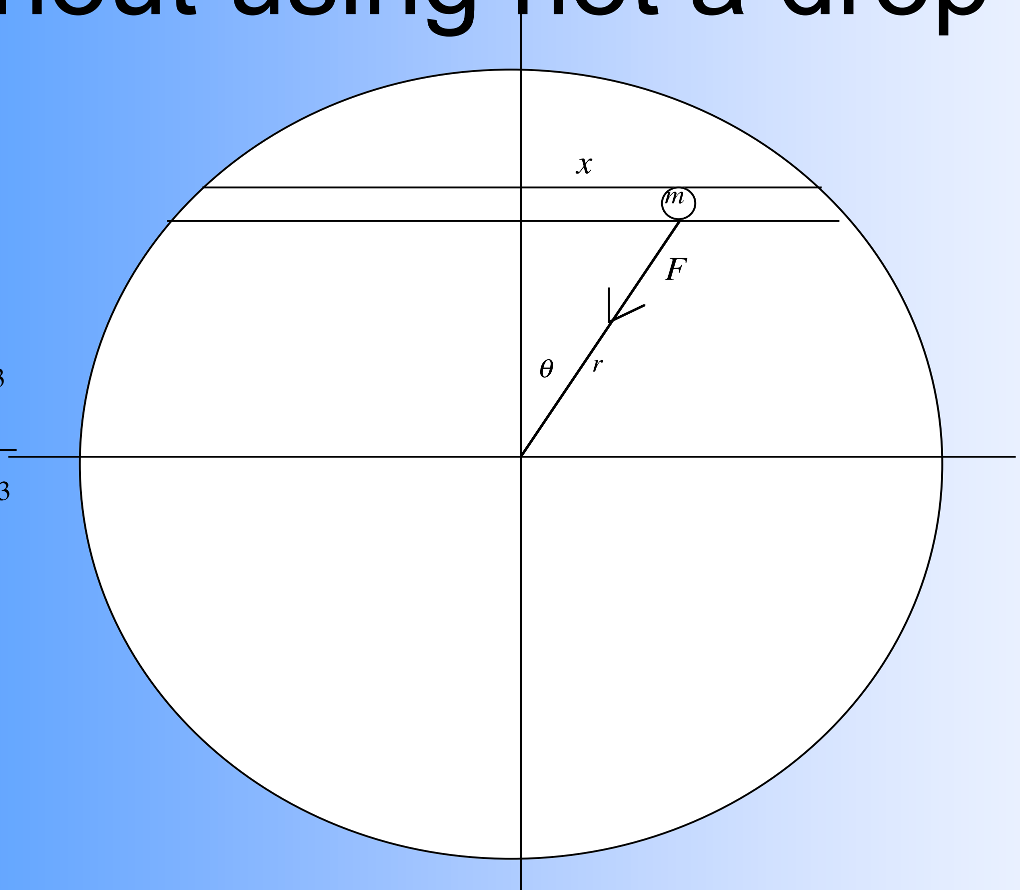


San Francisco to New York . . .
. . . in 45 minutes--
without using not a drop of gas!

$$\frac{M'}{M} = \frac{V'}{V}$$

$$\frac{M'}{M} = \frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi R^3}$$

$$M' = \frac{r^3}{R^3} M$$



$$F_x = -\frac{GmM'}{r^2}\sin\theta$$

$$= -\frac{GmM'}{r^2}\frac{x}{r}$$

$$= -\frac{GMm}{R^3}x$$

$$= -\frac{mg}{R}x$$

$$= -kx$$

$$T = 2\pi\sqrt{\frac{R}{g}}$$

To See and Do

1) Sketch a plot of g vs. r out to a distance of $5r$.

To See and Do

2) Use the values stated for *R* and *g* to calculate the *period* of the satellite in seconds, then in minutes. Write down the formula first, *then* plug in the numbers. Show your calculations.

To See and Do

3) Use the values stated for *R* and *g* to calculate the *velocity* of the satellite in orbit as shown in equation (7) in seconds, then in minutes. Write down the formula first, then plug in the numbers. Show your calculations.

To See and Do

4) Show at an altitude of 250 km (typical space shuttle orbit) that the acceleration of gravity *g* decreases to 92% of its value here at the surface of the earth.

To See and Do

5) Use the value of *g* calculated in #4 to calculate the *actual* period of a satellite with an orbit 250 km above the earth's surface seconds, then in minutes. How well does your value compare to oft quoted value of 90 minutes?

To See and Do

6) Suppose that boring a tunnel through the center of the earth proves too difficult, however, boring one from San Francisco to New York proves practical. How long would it take to go from one city to the other? Show your reasoning.

Assumptions

- In my analysis, I assumed the earth was spherical and of uniform density. As it turns out, of course, it isn't either.
- However, for the purposes of this discussion and first approximation, it is good enough for me. For a more complex analysis, I refer you to the following analysis by Dave Typinski:
- http://typnet.net/Essays/EarthGrav.htm